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DURATION GAP IN THE CONTEXT OF A BANK'S STRATEGIC PLANNING PROCESS

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Abstract

This paper presents an approach to duration that adds depth and realism to the subject of duration gap, which is usually presented as a "stand alone" issue in much of the banking literature. Duration is an important tool used by managers, but many overly simplified examples are not consistent with operating realities. This study offers a more realistic approach to measuring portfolio duration and duration gap which will enhance the bank's strategic planning process.

INTRODUCTION

While rates have remained relatively low and stable recently, institutions must still be prepared for rising interest rates. Duration matching represents a powerful tool in minimizing the risk of changing interest rates. It is an important tool utilized by the decision-makers in risk management issues. However, currently there are many illustrations that provide overly simplified definitions of portfolio duration and its use in immunization strategies to reduce interest rate risk. Unfortunately, many examples are not accurate or are not consistent with operating realities. This study offers a more realistic approach to measuring portfolio duration and duration gap. We offer illustrations correcting for some of the oversimplified examples of portfolio duration found in the literature. Further, we apply the corrected portfolio duration to an immunization example for a bank using duration gap.

Several authors have addressed the problem of oversimplification or mistakes in illustrations of duration and immunization. Bierwag and Kaufman [1] attempt to clarify duration gap for financial institutions by providing several single-factor duration gap equations. The authors explain that since immunization will depend on the account the institution targets as its primary concern, different duration gap equations should be adopted for alternative "target" accounts. While noting that Macaulay's duration is accurate in describing a security's price sensitivity *only* if the yield curve is flat and shifts in the yield curve are parallel, these authors choose to use this less realistic measure of duration so that they can focus on the differences in the duration gap measure resulting from the choice of desired target account.

Shaffer [11] also expresses concern over the restrictive condition of a flat yield curve in using Macaulay's duration and adds that changes in interest rates must be small for immunization to be effective. He concludes that these restrictive conditions are one reason why many financial institutions have been hesitant to adopt duration gap as a means of controlling interest rate risk. While Shaffer describes a perfectly hedged bank as one whose "duration of its assets, *weighted by dollars of assets*, equals the duration of its liabilities, *weighted by dollars of liabilities*," he notes that this is true only under simplifying assumptions.

Cherin and Hanson [2] examine the immunization strategy of matching the duration of a fixed income portfolio with investment horizon. They point out that most illustrations concentrate on the duration of principal payments but ignore the duration of accumulated interest payments. More specifically, these illustrations present a change in principal value when interest rates change (i.e., when interest rates decrease, the bond's market value increases), but do not show a similar change in the value of accumulated interest. The authors suggest that a more compatible treatment would be to assume both principal and accumulated interest are invested in bonds that are duration-matched to the investment horizon, therefore resulting in an increase in the values of both principal and accumulated

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interest when interest rates decline. They demonstrate, through an example, that applying this assumption and rebalancing the portfolio to maintain duration equal to the investment horizon will lead to complete immunization.

While the works by Bierwag and Kaufman [1], Cherin and Hanson [2], and Shaffer [11] sited above provide an excellent framework for analyzing the problem of risk-management faced by financial institutions, many fail to utilize this framework in a realistic fashion. In fact many bank related works portray duration gap as:[1]

> The net worth of any bank is equal to its assets less its liabilities. By equating duration of assets and duration of liabilities, a bank can immunize its net worth against changes in interest rates. The goal is to make the duration gap (duration of asset portfolio minus duration of bank liabilities) as close to zero as possible.

There are two flaws with the analysis for the strategic planner at financial institutions. First, this goal implies that a bank needs to ensure that the duration of the assets is equal to the duration of the liabilities multiplied by the ratio of the total liabilities to the total assets. However, our study will demonstrate that this simplified relationship implies some assumptions that are unrealistic to banking operations. In addition, these authors, along with Saunders [10] and Madura [7], define the duration of a portfolio as equaling the dollar weighted average of the duration of each individual asset in the portfolio. We will illustrate below that this is an accurate representation of portfolio duration only if the yields on the financial instruments are the same. Haugen [5] and Tuckman [13] provide additional illustrations.

PORTFOLIO DURATION

In some instances simplifying assumptions can cause confusion for the manager as he or she tries to reconcile the presentation of duration gap relative to bank operating realities. For example, one might question whether the yield and duration of a portfolio of fixed income instruments are actually equal to the dollar-weighted average of the yields and duration of each individual asset. If we examine this issue using two zero coupon bonds, we can demonstrate that the weighted average yield and duration of the individual assets are not necessarily equal to that of the portfolio as a whole. Zero coupon bonds are ideal for illustration, since Macaulay duration is equal to maturity for zero coupon bonds. Table 1 provides a comparison between current text illustrations of portfolio duration (simple weighted average duration) and the correct calculation of portfolio duration (actual portfolio duration). The simple weighted average duration provided in this table and the next (Table 2) are comparable to examples provided in literature directed at bank managers (including those cited in this paper). A one-year 6% yield to maturity (YTM) zero coupon bond and a ten year 12% YTM zero coupon bond are used in the illustration. The simple weighted average yield and duration are 9% and 5.5 years, respectively, while the portfolio yield and the Macaulay duration of the portfolio are 11.47% and 5.7 years. The modified duration for the simple weighted average and portfolio duration are 5.05 and 5.11, respectively.

TABLE 1
Portfolio Duration Example
Simple versus Actual Portfolio Duration

Summary:	Simple Weighted Average Duration (in current texts)	Actual Portfolio Duration (suggested by authors)
Yield	9%	11.47%
Macaulay Duration	5.5 years	5.7 years
Modified Duration	5.05	5.11

TABLE 1
Portfolio Duration Example
Simple versus Actual Portfolio Duration
(Cont'd)

Security 1 1 year zero coupon bond, yield 6%, current market price[8] = $1,000, maturity value in one year = $1,060.00
Security 2 10 year zero coupon bond, yield 12%, current market price = $1,000, maturity value in ten years = $3,105.85

Present value of the portfolio = $1,000 + $1,000 = $2,000

Simplified Versions in Current Texts

Simple weighted average yield = .5(6%) + .5(12%) = 9%
Simple weighted average Macaulay duration = .5(1 year) + .5(10 years) = 5.5 years
Simple weighted average modified[9] duration = 5.05

Correct Version

Actual Portfolio yield and duration:

$$\text{Price}_{\text{portfolio}} = \frac{CF_1}{(1+y)^1} + \frac{CF_2}{(1+y)^2} + \ldots\ldots + \frac{CF_9}{(1+y)^9} + \frac{CF_{10}}{(1+y)^{10}}$$

$$\$2{,}000 = \frac{1060}{(1+y)^1} + \frac{0}{(1+y)^2} + \ldots\ldots + \frac{0}{(1+y)^9} + \frac{3105.85}{(1+y)^{10}}$$

$\text{Yield}_{\text{portfolio}} = 11.47\%$

$$\text{Macaulay Duration}_{\text{portfolio}} = \frac{\$950.93(1) + \$1{,}048.58(10)}{\$2{,}000} = 5.7 \text{ years}$$

$\text{Modified Duration}_{\text{portfolio}} = 5.11$

If we expand the illustration presented in Table 1 to take into account the fact that a bank's asset portfolio would be represented by a diverse group of investments, the disparity would be equally pronounced. Table 2A provides a simplified hypothetical bank balance sheet. This balance sheet is similar to examples provided in current texts. The simple weighted average yield and modified duration for the assets of the balance sheet (Table 2A) are calculated to be 9.75% and 5.2, respectively. The yield and duration that describes the bank's portfolio would in fact be determined by the portfolio's cash flows. Given the value of the portfolio and its periodic cash flows illustrated in Table 2B, the overall portfolio yield is found to be 10.22% and the modified duration is 4.90. This represents a significant difference from the simple weighted average.

TABLE 2
Bank Asset Portfolio
Simple versus Actual Portfolio Duration

Summary:	Simple Weighted Average	Actual Portfolio
Yield	9.75%	10.22%
Modified Duration	5.20	4.90

TABLE 2A: Simple Weighted Average Duration

Asset	Yield to Maturity	Investment	Modified Duration
1 year T-bill	5%	$ 1 million	.95
2 year installment loan (consumer)	14%	$ 1 million	1.30
10 year zero coupon strip	12%	$ 1 million	8.93
30 year fixed rate mortgage	8%	$ 1 million	9.44

Simple weighted average yield = .25(5% + 14% + 12% + 8%) = 9.75%
Simple weighted average modified duration = .25(.95 + 1.30 + 8.93 + 9.44) = 5.2

TABLE 2B: Actual Portfolio Duration

Asset[10]	Present Value	Y_1	Y_2 …	Y_{10} …	Y_{30}
1 year T-bill	**$ 1 million**	1,050,000	0	0	0
2 year installment loan (consumer)	**$ 1 million**	607,290	607,290		
10 year zero coupon strip	**$ 1 million**	0	0	3,105,848	0
30 year fixed rate mortgage	**$ 1 million**	88,827	88,827	88,827	88,827

$$\$\ 4{,}000{,}000 = \frac{1{,}746{,}117}{(1+y)^1} + \frac{696{,}117}{(1+y)^2} + \frac{88{,}827}{(1+y)^3} + \ldots + \frac{88{,}827}{(1+y)^9} + \frac{3{,}194{,}675}{(1+y)^{10}} + \ldots + \frac{88{,}827}{(1+y)^{30}}$$

$\text{Yield}_{\text{portfolio}}$ = 10.22%
$\text{Macaulay Duration}_{\text{portfolio}}$ = 5.4 years
$\text{Modified Duration}_{\text{portfolio}}$ = 4.90

DURATION GAP

Once we have attained the correct duration of the bank's asset portfolio, one must determine the duration of the bank's liabilities in order to compute the bank's duration gap. If we assume that the bank desires a duration gap equal to zero, then according to some prevailing bank texts, its capital will be immunized when the duration of the assets is equal to the duration of the liabilities (D_A=D_L). The only necessary adjustment being the application of the ratio of bank liabilities to total assets such that a duration gap equal to zero is attained when:

Equation 1

$$D_A = D_L\left(\frac{TotalLiabilities}{TotalAssets}\right) \text{ or } D_L = D_A\left(\frac{TotalAssets}{TotalLiabilities}\right).$$

As was the case with the prevailing use of the simple weighted average, this simplification may also lead to confusion. If one views duration as an elasticity where

Equation 2

$$D = Elasticity = \frac{\Delta P / P}{\Delta r / (1+r)},$$

then it can be shown in Equation 3 below that:[2]

Equation 3

$$\Delta P = \frac{-\Delta r P D}{(1+r)},$$

where r = market rate of interest, P = price or principal amount, and D = duration. Using the accounting identity A = L + C, where A = assets, L = liabilities, and C = capital, one can show that for the bank's capital to be immunized, the change in owner's equity, C, must equal zero when rates change. For this to occur for any given change in r, the change in the value of the bank's assets, A, must equal the change in the bank's liabilities, L.

Using Equation 3 we can equate the change in the value of the bank's assets relative to its liabilities in the following way:

Equation 4

$$\Delta P_A = \frac{-\Delta r_A P_A D_A}{(1+r_A)} = \frac{-\Delta r_L P_L D_L}{(1+r_L)} = \Delta P_L,$$

where the *A* subscript represents assets and the *L* subscript represents liabilities. While this relationship is algebraically correct, it unfortunately does not address the specific operating characteristics that are essential to bank operations. From this expression the change in the dollar value of the assets (P_A) equals the change in the dollar value of the liabilities (P_L) when the duration of the assets (D_A) equals the duration of the liabilities (D_L) if the following assumptions hold true:

1. The rate of return on the assets (r_A) is equal to the rate of return on the liability (r_L).
2. The dollar value of the assets (P_A) is equal to the dollar value of the liability (P_L) used to fund the asset.
3. The change in the rate of interest in the asset market (r_A) is equal to the change in the rate of interest in the liability market (r_L).

However, the above assumptions are in direct conflict with basic bank operations. For example, the average cost of the bank's liabilities should be less than the overall yield on the bank's assets. Therefore in reality, $r_A > r_L$. If we relax assumption (1) and incorporate the inequality of r_A and r_L into Equation 4, C = 0 is achieved when:

Equation 5

$$D_L = \frac{D_A(1+r_L)}{(1+r_A)}.$$ [3]

If we relax the second assumption, one can state that in at least some cases, in order to fund the bank's assets, the dollar amount of the liabilities will in fact have to vary from that of the funded asset. With the current risk-based capital adequacy standards, the amount of capital and therefore liabilities necessary to fund those assets may differ significantly given the risk nature of the assets and the reserves required on the liability. Clouse, D'Antonio, and Fluck [3] present an approach for dealing with capital adequacy in a dynamic framework. Currently, the simple dollar volume of the assets does not determine the amount of capital and therefore liabilities used to fund the asset portfolio. Consequently, P_A will not equal P_L. Given the fact that the bank can determine its capital adequacy needs and Federal Deposit Insurance Corporation (FDIC) premiums and reserve requirements, a more accurate adjustment would be to take them into account directly and not simply use the ratio of total liabilities to total assets, as is so often shown to be the case.

Relaxing assumption (2) leads to a further adjustment to Equation 4. Let N represent the portion of the bank's liabilities necessary to fund non-earning assets and other obligations. When capital adequacy dictates that equity capital would exceed the level of the bank's fixed assets (i.e. premises), N may be negative. Therefore P_A, the dollar value of the assets, can always be expressed in terms of P_L, the dollar value of the liabilities:

Equation 6

$$P_A = (1 - N)P_L \, .$$

Taking the realities of funds management into account would indicate that C=0 when:

Equation 7

$$D_L = \frac{D_A(1-N)(1+r_L)}{(1+r_A)} \, .$$[4]

Since bank managers are well aware that the asset and liability mix of the bank is comprised of instruments from different markets, the change in interest rates across those markets most likely will not be the same. The final adjustment takes into account varying interest rate changes across the asset and liability markets. This allows one to express the duration of the liability mix necessary to immunize the bank's capital in terms of the duration of the asset mix. Mathematically it is shown to be:

Equation 8

$$D_L = \frac{D_A(1-N)(1+r_L)(\Delta r_A)}{(1+r_A)(\Delta r_A)} \, ,$$

where r_A is the portfolio yield and r_L is the portfolio cost.[5]

EXAMPLE

At this point, a *simple* example might be useful in illustrating the appropriate adjustment. In order to minimize the complexity of our illustration, we do not incorporate Cherin and Hanson's [2] concept of including the duration of accumulated interest payments. Table 3A shows a bank balance sheet in which the simple weighted average modified duration of assets is 4.99 and of liabilities is 3.98. The simple weighted average yield for the assets portfolio is 7.80%, while the simple weighted average cost for the liabilities is 7.33%.[6]

Table 3B computes the actual duration for the portfolio of assets and liabilities. Both differ from the simple weighted average duration from Table 3A. The modified duration of the assets is 5.22, and the modified duration of the liabilities is 4.18. The actual portfolio yield (8.82%) and cost (8.47%) from Table 3B also differ from the simple weighted average yield (7.80%) and simple weighted average cost (7.33%) in Table 3A.

Table 4 determines the duration of the bank's liabilities required so that the bank's capital is immunized. First, we use the simplified Equation 1 with the simple weighted average modified duration of assets of 4.99 from Table 3A to calculate the required modified duration of liabilities of 5.66. Under the more realistic relationship, we use Equation 8 to compute the required liability duration. We then explore two cases. The first case assumes N, the portion of the bank's liabilities necessary to fund non-earning assets, is equal to zero. This allows P_A to equal P_L. The second case uses a more realistic assumption of N equal to three percent. In both cases, we consider the same varying rate changes across the asset and liability markets, or r_L=1% and r_A=0.5%. The computed duration of liabilities is significantly smaller under the more realistic relationship developed in Equation 8: 2.6 for case (1) and 2.52 for case (2). This disparity will have significant consequences for a bank's capital in an environment of changing interest rates. Equation 1 would not lead to a duration gap equal to zero and therefore the bank's capital would not be immunized.[7]

TABLE 3
A Simple Banking Example
Simple versus Actual Portfolio Duration

Summary:	Simple Weighted Average	Actual Portfolio
Assets		
Yield	7.80%	8.82%
Modified Duration	4.99	5.22
Liabilities		
Yield	7.33%	8.47%
Modified Duration	3.98	4.18

TABLE 3A: Simple Weighted Average Duration

Assets	**Market Value**	**Yield / Cost**	**Modified Duration**
Cash and Due	$ .5 million	0%	0
1 year US Treasury Bills	10.25 million	5%	.95
2 year Commercial Installment Loans	10.25 million	9%	1.36
20 year US Treasury Bond	10.25 million	8%	9.82
30 year Real Estate Loans	10.25 million	10%	8.34
Bank Premises	.5 million	0%	0
Total Assets	**$42 million**		

Simple weighted average yield = (10/41) x (5% + 9% + 8% + 10%) = 7.80%

Weighted average modified duration of assets = (10/41) x (.95 + 1.36 + 9.82 + 8.34) = 4.99

Liabilities	**Market Value**	**Yield / Cost**	**Modified Duration**
Demand Deposits	$ 1 million	0%	0
1 year Negotiable CD[11]	12 million	6%	.94
2 year Other Time Deposits[11]	12 million	7%	1.87
20 year Subordinated Notes	12 million	9%	9.13
Total Liabilities	**$37 million**		
Stockholder's Equity	5 million		
Total Liabilities and Stockholder's Equity	**$42 million**		

Simple weighted average cost = (12/36) x (6% + 7% + 9%) = 7.33%

Weighted average modified duration of liabilities = (12/36) x (.94 + 1.87 + 9.13) = 3.98

TABLE 3B: Actual Portfolio Duration

Cash Flows (Assets)	Y_1	Y_2 …	Y_5 …	Y_{20} …	Y_{30}
$.5 million cash and due					
$10.25 million 1 year T bills	$10,726,500				
$10.25 million 2 year Commercial Loan	5,826,806	$5,826,806 …			
$10.25 million 20 year Treasury Bond	820,000	820,000 …	$ 820,000 …	$11,070,000 …	
$10.25 million 30 year Real Estate Loan	1,087,312	1,087,312 …	1,087,312 …	1,087,312 …	$1,087,312
$.5 million Bank Premises					

$$\$41{,}000{,}000 = \frac{18{,}496{,}618}{(1+y)^{1}} + \frac{7{,}734{,}118}{(1+y)^{2}} + \frac{1{,}907{,}312}{(1+y)^{3}} + \cdots \frac{1{,}907{,}312}{(1+y)^{19}} + \frac{12{,}157{,}312}{(1+y)^{20}} + \frac{1{,}087{,}312}{(1+y)^{21}} \cdots + \frac{1{,}087{,}312}{(1+y)^{30}}$$

Portfolio Yield = 8.82%
Portfolio Modified Duration = 5.22

Cash Flows (Liabilities)	Y_1	Y_2 …	Y_5 …	Y_{20}
$1 million Demand Deposits				
$12 million 1 year Neg. CD	$12,720,000			
$12 million 2 year Other Time Deposits		$13,738,800 …		
$12 million 20 year Subordinated Notes	1,080,000	1,080,000 …	$1,080,000 …	

$$\$36{,}000{,}000 = \frac{13{,}800{,}000}{(1+y)^{1}} + \frac{14{,}818{,}800}{(1+y)^{2}} + \frac{1{,}080{,}000}{(1+y)^{3}} + \cdots + \frac{1{,}080{,}000}{(1+y)^{19}} + \frac{13{,}080{,}000}{(1+y)^{20}}$$

Portfolio Cost = 8.47%
Portfolio Modified Duration = 4.18

CONCLUSION

This paper demonstrates that the yield and duration of a portfolio of fixed income instruments are not necessarily equal to the weighted average of the yields and duration of each individual asset. In addition, we directly integrate both funds management and capital adequacy with portfolio duration. Our approach not only uses duration matching to minimize interest rate risk, it allows: (1) the rate of return on assets to exceed the rate of return on liabilities, and (2) the dollar value of the assets to differ from the dollar value of the liabilities used to fund those assets. Consequently, we address the major realities facing bank managers when making their strategic decisions.

This approach provides a touch of realism to the process of bank management as opposed to dealing with duration gap as a "stand alone" issue. Many examples do not address these realities and may therefore lead to misleading results. As a consequence, the banks strategic planning process may be misguided as a result of focusing on inappropriate goals and targets.

TABLE 4
Immunization Example

Using Equation 1, the simplified relationship is:

$$D_L = D_A\left(\frac{TotalAssets}{TotalLiabilities}\right) = 4.99\left(\frac{\$42\,million}{\$37\,million}\right) \qquad = \mathbf{5.66}$$

where D_A is the weighted average modified duration of assets.

Using Equation 8, the more realistic relationship is:

$$D_L = \frac{D_A(1-N)(1+r_L)(\Delta r_A)}{(\Delta r_L)(1+r_A)},$$

where D_A is the portfolio modified duration.

<u>Case (1):</u> N=0, r_L=1%, and r_A=0.5%

$$D_L = \frac{5.22(1-0)(1+.0847)(.005)}{(.01)(1+.0882)} \qquad = \mathbf{2.60}$$

<u>Case (2):</u> N=3%, r_L=1%, r_A=0.5%

$$D_L = \frac{5.22(1-.03)(1+.0847)(.005)}{(.01)(1+.0882)} \qquad = \mathbf{2.52}$$

ENDNOTES

1. Examples include texts by Rose [8], Thygerson [11], Mishkin and Eakins [7], Koch [5], and Gardner and Mills [3].
2. See Rose [8].
3. The algebraic derivation of Equation 5 from Equation 4 and assumptions (2) and (3) is shown in Appendix A.
4. The derivation of Equation 7 from Equations 4 and 6 and assumption (3) is shown in Appendix A.
5. The derivation of Equation 8 from Equations 4 and 6 is shown in Appendix A.
6. Calculations are provided in Appendix B.
7. See Appendix C.
8. The prices of the bonds are both $1,000 to illustrate the simple weighted average yield and duration clearly with 50% weights. Normally, zeros would sell at a discount and have maturity value of $1,000.
9. Modified duration is calculated as the Macaulay duration divided by (1+YTM).
10. The prices, or present values, of the bonds are chosen to preserve equal weights in the portfolio. The one-year Treasury bill, for example, would actually be sold at a discount and have maturity value of $1 million.
11. For simplicity, interest is assumed to be paid at the end of the period.

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APPENDIX A
Algebraic Derivations for Equations 5, 7, and 8

From Equation 4: $$\frac{-\Delta r_A P_A D_A}{(1+r_A)} = \frac{-\Delta r_L P_L D_L}{(1+r_L)}$$

If $P_A = P_L$ (assumption 2), then $$\frac{-\Delta r_A P_A D_A}{(1+r_A)} = \frac{-\Delta r_L P_A D_L}{(1+r_L)}$$

$$\frac{\Delta r_A D_A}{(1+r_A)} = \frac{\Delta r_L D_L}{(1+r_L)}$$

$$D_L = \frac{\Delta r_A D_A (1+r_L)}{\Delta r_L (1+r_A)}.$$

If $r_A =$ r_L (assumption 3), then $$D_L = \frac{D_A (1+r_L)}{(1+r_A)}.$$ **Equation 5**

Since $r_A > r_L$, we cannot reduce the equation further.

From Equation 4: $$\frac{-\Delta r_A P_A D_A}{(1+r_A)} = \frac{-\Delta r_L P_L D_L}{(1+r_L)}$$

If $r_A =$ r_L (assumption 3), then $$\frac{P_A D_A}{(1+r_A)} = \frac{P_L D_L}{(1+r_L)}$$

$$D_L = \frac{P_A D_A (1+r_L)}{P_L (1+r_A)}.$$

From Equation 6: $P_A = (1 - N)P_L$

$$D_L = \frac{(1-N) P_L D_A (1+r_L)}{P_L (1+r_A)}$$

$$D_L = \frac{D_A (1-N)(1+r_L)}{P_L (1+r_A)}.$$ **Equation 7**

From Equation 4: $$\frac{-\Delta r_A P_A D_A}{(1+r_A)} = \frac{-\Delta r_L P_L D_L}{(1+r_L)}$$

From Equation 6: $P_A = (1 - N)P_L$

$$\frac{-\Delta r_A (1-N) P_L D_A}{(1+r_A)} = \frac{-\Delta r_L P_L D_L}{(1+r_L)}$$

$$D_L = \frac{-\Delta r_A (1-N) P_L D_A (1+r_L)}{-\Delta r_L P_L (1+r_A)}$$

$$D_L = \frac{D_A (1-N)(1+r_L)\Delta r_A}{(1+r_A)\Delta r_L}$$ **Equation 8**

APPENDIX B
Calculations for Table 4 (in $10,000)

Assets

2 year Commercial Installment Loan: 9%, 2 equal payments

Year	Payment	Present Value Factor		
Y_1	$582.68	.917	(1) =	$ 534.57
Y_2	582.68	.842	(2) =	980.86
				$1,515.43

$$Macaulay's Duration = \frac{1,515.55}{1,025} = 1.48\,years$$

$$Modified Duration = \frac{1.48}{1.09} = 1.36$$

20 year 8% Treasury Bond

Year	Payment	Present Value Factor				
Y_1	$ 82	.926	=	74.00	(1) =	$ 75.93
Y_2	82	.857	=	68.56	(2) =	140.60
Y_3	82	.794	=	63.52	(3) =	195.28
Y_4	82	.735	=	58.80	(4) =	241.09
Y_5	82	.681	=	54.48	(5) =	279.04
Y_6	82	.630	=	50.40	(6) =	310.04
Y_7	82	.583	=	46.64	(7) =	334.92
Y_8	82	.540	=	43.20	(8) =	354.42
Y_9	82	.500	=	40.00	(9) =	369.18
Y_{10}	82	.463	=	37.04	(10) =	379.82
Y_{11}	82	.429	=	34.32	(11) =	386.85
Y_{12}	82	.397	=	31.76	(12) =	390.76
Y_{13}	82	.368	=	29.44	(13) =	391.97
Y_{14}	82	.340	=	27.20	(14) =	390.85
Y_{15}	82	.315	=	25.20	(15) =	387.75
Y_{16}	82	.292	=	23.36	(16) =	382.96
Y_{17}	82	.270	=	21.60	(17) =	376.75
Y_{18}	82	.250	=	20.00	(18) =	369.37
Y_{19}	82	.232	=	18.56	(19) =	361.00
Y_{20}	1,107	.215	=	232.20	(20) =	4,750.10
				1,000.00		$10,868.69

$$Macaulay's Duration = \frac{10,868.69}{1,025} = 10.60\,years$$

$$Modified Duration = \frac{10.60}{1.08} = 9.82$$

APPENDIX B
Calculations for Table 4 (in $10,000)

Assets

30 year Real Estate Loan: 10%, 30 equal payments

Year	Payment	Present Value Factor			
Y_1	$108.73	.909	(1)	=	$ 98.85
Y_2	108.73	.826	(2)	=	179.72
Y_3	108.73	.751	(3)	=	245.07
Y_4	108.73	.683	(4)	=	297.06
Y_5	108.73	.621	(5)	=	337.57
Y_6	108.73	.564	(6)	=	368.26
Y_7	108.73	.513	(7)	=	390.57
Y_8	108.73	.467	(8)	=	405.79
Y_9	108.73	.424	(9)	=	415.01
Y_{10}	108.73	.386	(10)	=	419.21
Y_{11}	108.73	.350	(11)	=	419.21
Y_{12}	108.73	.319	(12)	=	415.74
Y_{13}	108.73	.290	(13)	=	409.44
Y_{14}	108.73	.263	(14)	=	400.85
Y_{15}	108.73	.239	(15)	=	390.44
Y_{16}	108.73	.218	(16)	=	378.61
Y_{17}	108.73	.198	(17)	=	365.70
Y_{18}	108.73	.180	(18)	=	352.01
Y_{19}	108.73	.164	(19)	=	337.79
Y_{20}	108.73	.149	(20)	=	323.24
Y_{21}	108.73	.135	(21)	=	308.55
Y_{22}	108.73	.123	(22)	=	293.86
Y_{23}	108.73	.112	(23)	=	279.29
Y_{24}	108.73	.102	(24)	=	264.94
Y_{25}	108.73	.092	(25)	=	250.89
Y_{26}	108.73	.084	(26)	=	237.20
Y_{27}	108.73	.076	(27)	=	223.93
Y_{28}	108.73	.069	(28)	=	211.11
Y_{29}	108.73	.063	(29)	=	198.78
Y_{30}	108.73	.057	(30)	=	186.94
					$9,405.63

$$Macaulay's\,Duration = \frac{9{,}405.63}{1{,}025} = 9.18\,years$$

$$Modified\,Duration = \frac{9.18}{1.10} = 8.35$$

APPENDIX B
Calculations for Table 4 (in $10,000)

Liabilities

20 year 9% Subordinated Notes

Year	Payment	Present Value Factor			
Y_1	$ 135	.917	(1)	=	$ 123.80
Y_2	135	.842	(2)	=	227.34
Y_3	135	.772	(3)	=	312.66
Y_4	135	.708	(4)	=	382.32
Y_5	135	.650	(5)	=	438.75
Y_6	135	.596	(6)	=	482.76
Y_7	135	.547	(7)	=	516.92
Y_8	135	.502	(8)	=	542.16
Y_9	135	.460	(9)	=	558.90
Y_{10}	135	.422	(10)	=	569.70
Y_{11}	135	.388	(11)	=	576.18
Y_{12}	135	.356	(12)	=	576.72
Y_{13}	135	.326	(13)	=	572.13
Y_{14}	135	.299	(14)	=	565.11
Y_{15}	135	.275	(15)	=	556.88
Y_{16}	135	.252	(16)	=	544.32
Y_{17}	135	.231	(17)	=	530.15
Y_{18}	135	.212	(18)	=	515.16
Y_{19}	135	.194	(19)	=	497.61
Y_{20}	1,635	.178	(20)	=	5,820.60
					14,925.17

$$Macaulay's\,Duration = \frac{14{,}925.17}{1{,}500} = 9.95\,years$$

$$Modified\,Duration = \frac{9.95}{1.09} = 9.13$$

APPENDIX C
Overall Asset and Liability Mix

Tables 4 and 5 show computations for yields and durations without including the *Cash and Due* and *Bank Premises* in the assets and *Demand Deposits* in the liabilities. Since these items are not interest sensitive on a continuous basis, one could argue that duration is not defined for these items. If duration for these items were included in the calculations and simply given a value of zero, the results would be as follows:

	Assets	Liabilities
Simple weighted average yield	7.62%	7.14%
Simple weighted average modified duration	4.86	3.87
Portfolio yield	8.36%	7.84%
Portfolio modified duration	5.42	4.39

Using the overall asset and liability mix, we find an even wider disparity between the simple weighted average duration and the portfolio durations. Using the numbers above, the immunization example in Table 6 would show a wider discrepancy for the desired duration of liabilities between the simplified relationship implied from Equation 1 of 5.52 and the more realistic relationship implied from Equation 8 of 2.70 (under case 1).