# Valuation of Lease Contracts In Continuous Time With Stochastic Asset Values

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**Abstract**

*A lease is a derivative security the value of which depends upon the value of the underlying asset. Using contingent-claim analysis, this paper evaluates different lease contracts. The value of the lease depends upon the options embedded in the lease contract. Results for three special cases are derived and illustrated when it is assumed that the value of the underlying asset decreases linearly with time but in a stochastic environment.*

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*JEL Classification: G12, G13, G32*

## 1. Introduction

A considerable amount of work has been done on the analysis of leasing as a financial decision of the firm. A comprehensive paper by Myers, Dill, and Bautista [1976] discusses the fundamental issues in leasing and presents a thorough analysis of the valuation of leasing contracts based ~~on~~ upon fundamental financial principles. Furthermore, it as been shown that leasing can have an impact on the capital structure of the firm, particularly on its debt capacity [Bowman, 1980]. Miller and Upton [1976] discuss the cost of capital associated with leasing. McConnell and Schallheim [1983] use option-pricing theory to evaluate a variety of options embedded in lease contracts. They discuss the value of cancelable and non-cancelable contracts, the option that gives the right to the lessee to extend the life of the lease, and the option that the lessee can purchase the asset at the end of the lease, either at the fair market value or at a fixed price. The discussion of both papers, however, is limited to asset values that follow *geometric* Brownian motion.

[+] I would like to acknowledge the helpful comments of the anonymous referee. The remaining errors are my own.

The present paper extends the work by McConnell and Schallheim by assuming that the value of the asset follows *arithmetic* Brownian motion. This implies that the lease value is falling linearly with time, not exponentially. This is a more appropriate assumption in the case of some industrial and construction equipment. For instance, the value of a tractor is depends on the number of hours it has been operating as shown by its time-meter.[1]

Grenadier [1995, 2005] also present a thorough analysis of the valuation of options embedded in lease contracts based on fundamental economic principles. Another paper by Grenadier [1996] analyzes the credit risk associated with leasing. However, unlike the previous analyses, the present paper uses *arithmetic* Brownian motion, which is more relevant for many leased assets. This paper will then discuss three special cases of leasing which can be used to evaluate the special cases of automobile and real estate leases.

## 2. Lease Contracts in Continuous Time

As background, consider the theory of lease valuation in discrete time. When the owner of an asset (lessor) and the prospective user of the asset (lessee) agree upon the conditions under which the latter may use the asset, they set the terms in a legal contract called the lease. The value of the lease to the lessor depends on the lease payments received from the lessee and the expected residual value of the asset at the termination of the lease. Therefore, the value of a lease in discrete time may be written as

$$V = \sum_{i=1}^{T} \frac{p(1 - t) + tD}{(1 + r)^i} + \frac{x_T}{(1 + r)^T} \tag{1}$$

where $V$ = the value of the lease to the lessor
$p$ = the periodic lease payments received by the lessor per unit time
$t$ = the income tax rate of the lessor
$D$ = the depreciation of the asset per unit time
$r$ = the relevant discount rate, which may be the risk-free rate if the cash flows from the lease are guaranteed and the final value of the asset is known with certainty. This may be the after-tax cost of borrowing by the lessee.

$x_T$ = the residual value of the asset at the termination of the lease
$T$ = the length of the lease in units of time.

If the income tax rate of the lessor is zero, then (1) becomes

$$V = \sum_{i=1}^{T} \frac{p}{(1 + r)^i} + \frac{x_T}{(1 + r)^T} \tag{2}$$

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[1]One can see an online listing of tractors at http://www.tractorhouse.com/listings/forsale. It lists thousands of used tractors for sale with their cumulative hours of operation and prices. It is possible to narrow the search to a particular manufacturer, model, and year. In that subset, the tractors generally show a decrease in value for increasing total number of hours of operation.

In continuous-time finance, however, (2) may be written as

$$V = \int_0^T p\, e^{-rt}\, dt + x_T\, e^{-rT} \tag{3}$$

where $p$ is the lease payment per period, received continuously, and $r$ is the continuously compounded appropriate discount rate. Simplifying (3), results in

$$V(x,T) = \frac{p\,(1 - e^{-rT})}{r} + x_T\, e^{-rT} \tag{4}$$

where $x$ is the present value of the asset and $T$ is the length of the time that the lease is in force. The residual value $x_T$ of the asset is assumed to decrease linearly with time, and may be expressed as

$$x_T = x - \alpha T \tag{5}$$

The value of the lease $V$ then becomes

$$V(x,T;\alpha,r,p) = \frac{p\,(1 - e^{-rT})}{r} + (x - \alpha T)e^{-rT}, \quad 0 < T < T_0 \tag{6}$$

The maximum value of the lease is for time $T_0 = x/\alpha$. Substituting this value of $T$ in (6) results in

$$V(r,T_0) = \frac{p\,(1 - e^{-rx/\alpha})}{r} \tag{7}$$

The above equation represents the value of a lease which remains in effect as long as the asset is working and when the value of the asset is declining in a *deterministic* way.

However, when the asset value is decreasing in a *stochastic* manner the following simplifying assumptions are made.

1. There are no taxes and no transaction costs.
2. The asset values follow arithmetic Brownian motion. (This is the principal assumption.) The straight-line method of depreciation of the asset equals the true economic devaluation. The drop is not exactly uniform, and the value $x$ follows arithmetic Brownian motion, represented by the stochastic differential equation (8).

$$dx = -\,\alpha\, dt + \sigma\, dz \tag{8}$$

3. The drift factor $\alpha$ and the volatility $\sigma$ in the above equation are positive constants.
4. The interest rate $r$ remains constant, with a flat term structure.

In (8), $dx$ represents the change in $x$ in time $dt$, and $dz$ is the standard Wiener process. The drift parameter $\alpha$ and the variance parameter $\sigma$ are both positive. The value of the asset is gradually decreasing with time but with random fluctuations. When $x = 0$, there is an absorbing barrier, and the process stops. This means that once the value of the asset becomes zero, it cannot become positive again. At the beginning of the lease, when $t = 0$, the value of the asset is $I_0$.

By Itô's lemma, the change in $V(x,t)$ may be written as

$$dV = \frac{\partial V}{\partial x} dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial V}{\partial t} dt \tag{9}$$

Substituting from (8) and taking the expected value produces

$$\mathrm{E}(dV) = -\alpha \frac{\partial V}{\partial x} dt + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} dt + \frac{\partial V}{\partial t} dt \tag{10}$$

The above equation also represents the capital gain from the lease. The cash flow from the lease for $dt$ is

$$\mathrm{E}(\text{cash flow}) = p\,dt \tag{11}$$

Adding the capital gain to the cash flow gives the total return of the lease $V$ at the rate $r$ for the time $dt$ as

$$rV\,dt = -\alpha \frac{\partial V}{\partial x} dt + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} dt + \frac{\partial V}{\partial t} dt + p\,dt \tag{12}$$

If the values of $\alpha$ and $\sigma$ are known precisely and if the lessee makes guaranteed payments $p$, then $r$ may be the risk-free rate; otherwise, it is the pretax cost of debt for the corporation. Rearranging terms yields

$$\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - \alpha \frac{\partial V}{\partial x} - rV = -\frac{\partial V}{\partial t} - p \tag{13}$$

## 3. Case 1

Consider a lease wherein the lessee will use the asset until it becomes worthless and the lessor simultaneously receives a payment $p$ continuously per unit of time. The lease terminates when the value of the asset becomes zero. Under equilibrium conditions, the value of this lease should be equal to the value of the asset itself. If the lessor is able to charge lease payments in excess of the equilibrium value, the NPV of the lease will be positive for him. For the lessee, the lease contract eliminates uncertainty because the lessor will replace the asset if it fails.

In this case, there is no prearranged time for the termination of the lease. The asset will lose its value because of the negative drift term in (8), and it cannot be predicted when its value will fall to zero. The lease is like a perpetual security, except that the value of this security becomes zero at some indefinite time in the future.  For this reason, the value of the lease is no longer a function of time so that $V(x,t) = V(x)$. Then (13) becomes an ordinary differential equation

$$\frac{\sigma^2}{2}\frac{d^2V}{dx^2} - \alpha\frac{dV}{dx} - rV = -p \tag{14}$$

The solution is

$$V(x) = Ae^{n_1 x} + Be^{n_2 x} + p/r \tag{15}$$

where $$n_1 = \frac{\alpha + \sqrt{\alpha^2 + 2\sigma^2 r}}{\sigma^2} \quad \text{and} \quad n_2 = \frac{\alpha - \sqrt{\alpha^2 + 2\sigma^2 r}}{\sigma^2}\ . \tag{16}$$

The first boundary condition on the value of the lease requires that when the value of the asset becomes zero, the value of the lease also becomes zero,

$$V(0) = 0 \tag{17}$$

The second boundary condition requires that when the value of the underlying asset increases indefinitely, the rate of change of $V$ with respect to $x$ remains finite, or that

$$\left.\frac{dV}{dx}\right|_{x=\infty} < \infty \tag{18}$$

Imposing these boundary conditions gives $A = 0$ and $B = -p/r$.  Substituting the values of $A$ and $B$ in (15), gives the solution for Case 1 as

$$V_1(x) = \frac{p}{r}\left[1 - \exp\left(\frac{x[\alpha - \sqrt{\alpha^2 + 2\sigma^2 r}]}{\sigma^2}\right)\right] \tag{19}$$

For an asset with a very large value $x$, the value of a lease is similar to a perpetuity, thus

$$V(\infty) = p/r \tag{20}$$

The factor $n_2$ contains the random nature of $x$. With a very slow rate of depreciation in value, $\alpha << 1$, this case (16) would give

$$n_2 \cong -\frac{\sqrt{2r}}{\sigma} \tag{21}$$

The value of a lease with a slowly depreciating asset is therefore

$$V(x) = \frac{p}{r}\left[1 - \exp\left(\frac{-x\sqrt{2r}}{\sigma}\right)\right] \tag{22}$$

For an asset which shows very small fluctuations in its value, $n_2$ can be written as

$$n_2 = \lim_{\sigma \to 0} \frac{\alpha - \sqrt{\alpha^2 + 2\sigma^2 r}}{\sigma^2} = -\frac{r}{\alpha} \tag{23}$$

The value of a lease on such an asset is

$$V = \frac{p}{r}\left[1 - \exp\left(-\frac{rx}{\alpha}\right)\right] \tag{24}$$

This is the same as (7), the result obtained when the asset value was dropping uniformly.

Since the lease does not have a terminal date, time does not appear as a variable in (19), (20), and (22). Differentiating $V(p,x,r;\alpha,\sigma)$ from (19) with respect to the variables $p$, $x$, $r$; $\alpha$, and $\sigma$ produces

$$\frac{\partial V}{\partial p} = \frac{1 - \exp(n_2 x)}{r} > 0 \tag{25}$$

$$\frac{\partial V}{\partial x} = -\frac{p n_2 \exp(n_2 x)}{r} > 0 \tag{26}$$

$$\frac{\partial V}{\partial r} = -\frac{p[1 - \exp(n_2 x)]}{r^2} + \frac{px \exp(n_2 x)}{r\sqrt{\alpha^2 + 2\sigma^2 r}} < 0 \tag{27}$$

$$\frac{\partial V}{\partial \alpha} = \frac{px(\alpha - \sqrt{\alpha^2 + 2\sigma^2 r}) \exp(n_2 x)}{r\sigma^2 \sqrt{\alpha^2 + 2\sigma^2 r}} < 0 \tag{28}$$

$$\frac{\partial V}{\partial \sigma} = -\frac{2px(\sigma^2 r + \alpha^2 - \alpha\sqrt{\alpha^2 + 2\sigma^2 r}) \exp(n_2 x)}{r\sqrt{\alpha^2 + 2\sigma^2 r}} < 0 \tag{29}$$

To get a feel for these results, consider a numerical example in which the value of an asset is \$100 at the beginning of the lease, the expected rate of depreciation in value for this asset is \$20 per year with a standard deviation of \$5 per year$^{1/2}$, the relevant discount rate is 10% per annum, and the lease payments received continuously are \$25 per year. To find the value of this lease, let $x$ = \$100, $\alpha$ = \$20 per year, $\sigma$ = 5 per year$^{1/2}$, $p$ = 25 per year, and $r$ = 0.1 per year in (19). The result is \$98.13. If the initial investment in this asset was \$100, the net present value of the buy-and-lease decision ~~is~~ would be –\$1.87. For a lower discount rate, $r$ = 0.08, the net present value would be \$2.82.

The lease allows the lessee to continue using the asset until its value becomes zero. An asset with a higher initial value will then give a longer life to the lease. The other main factor that determines the value of the lease, the discount rate $r$, can be seen by plotting the value of the lease versus the discount rate for different values of the asset. Let $\alpha = 20$, $\sigma = 5$, $p = 25$, $r = 0$ to 0.2, and $x = 70$ to 100. Figure 1 displays these plots.

**Figure 1. Lease Value versus Discount Rate**

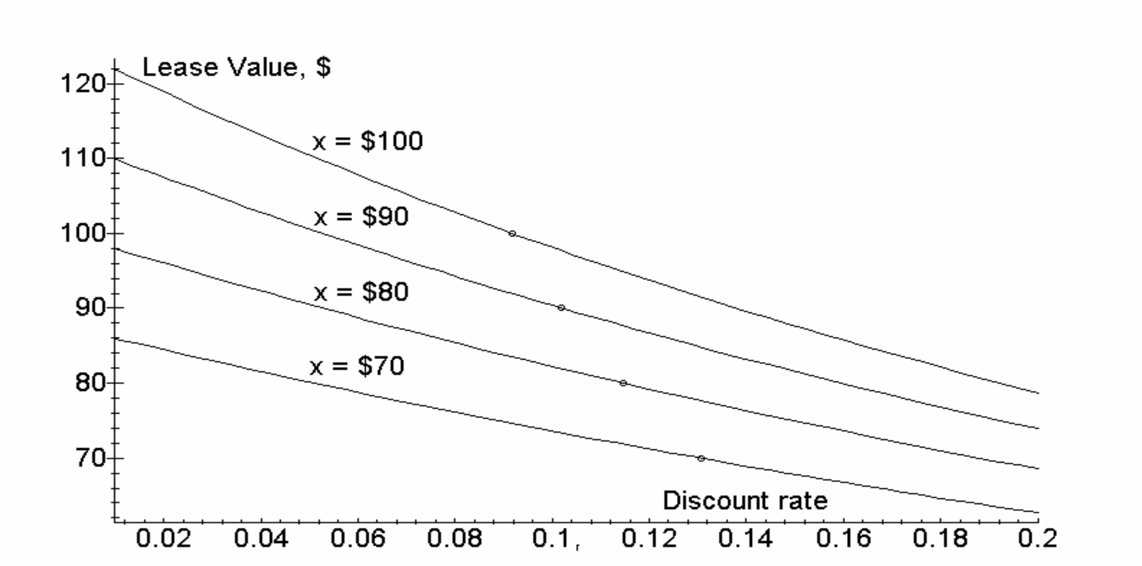


**Figure 1. The diagram above shows how the value of a lease with no time limit changes at various discount rates. The plots are for assets with initial values of \$100, \$90, \$80, and \$70. Other parameters are ά = 20, σ = 5, and *p* = 25. The small circles on the curves represent the points where the NPV of the lease is zero for the lessor.**

**Figure 2: Lease Values and Depreciation Rates**

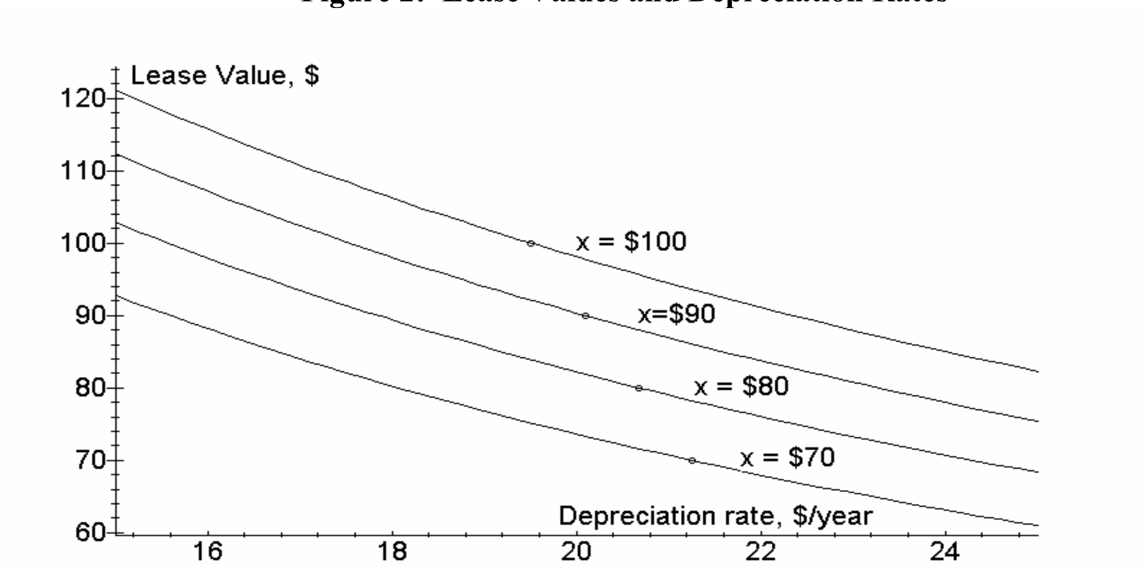


**Figure 2: The curves above show how the value of a lease with no time limit and with initial asset values of \$100, \$90, \$80, and \$70 varies with the depreciation rate ά. The other parameters are σ = 5, *p* = 25, and *r* = 0.1. The small circles on the curves show the points where the net present value of the lease is zero for the lessor.**

Figure 2 shows the effect of depreciation rate $\alpha$ on the lease value. Let the asset value $x$ = \$70 to \$100, $\sigma$ = \$5 per year$^{1/2}$, $p$ = \$25 per year, and $r$ = 0.1 per year. The plots show the drop in the value of the lease for an increase in the depreciation rate $\alpha$ from \$15 to \$30 per year. The small circles on the curves indicate the points where the NPV of the lease is zero for the owner of the asset.

**Figure 3: Lease Value versus Uncertainty of Depreciation**

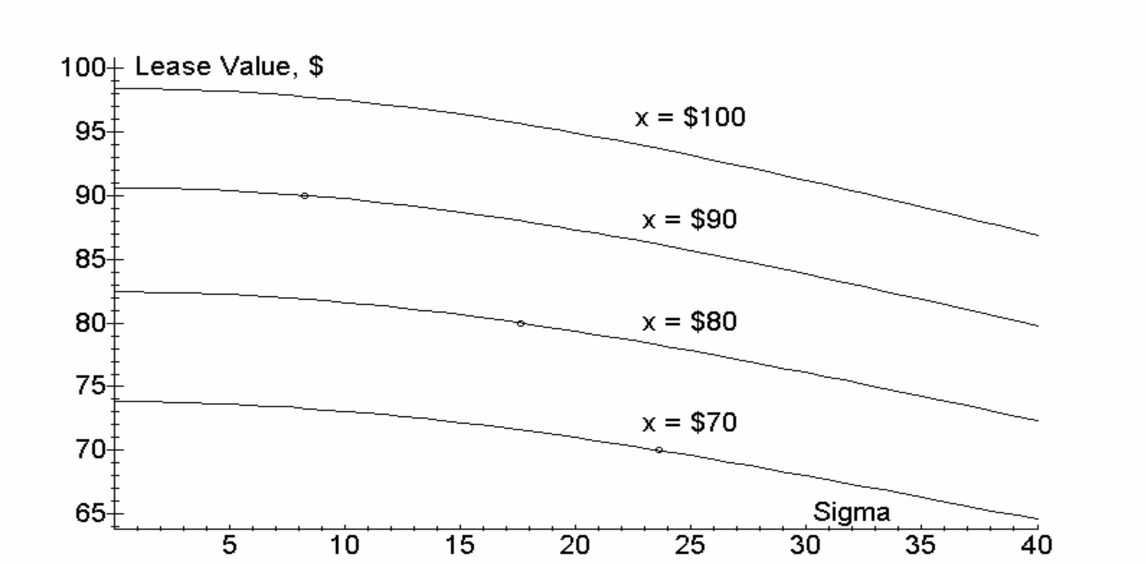


**Fig. 3: The diagram above shows how the value of a lease with no time limit and with initial values of \$100, \$90, \$80 and \$70 varies with $\sigma$ the standard deviation of the depreciation rate ά. ά = 20, and the other parameters are $p$ = 25, and $r$ = 0.1. The small circles on the curves represent the points where the net present value of the lease is zero for the lessor.**

Now compare assets with more predictable rates of depreciation (small $\sigma$) to those with greater uncertainty in their rate of depreciation (larger $\sigma$). Again, let $x$ = \$70 to \$100, $\alpha$ = \$20 per year, $p$ = \$25 per year, and $r$ = 0.1 per year, and allow $\sigma$ to vary from zero to \$40 per year$^{1/2}$. Figure 3 displays these curves. The lease value declines with greater $\sigma$ but at a rather slow rate, implying that the $\sigma$ of the asset does not play an important role in the lease valuation. Again, the small circles on the curves indicate the points where the net present value of the lease is zero for the lessor.

## 4. Case 2

In this case, the lease allows the lessee to use an asset as long as it is operating. The lessee pays $p$ per unit time continuously over time. The lease has a maturity time $T$. If the asset breaks down before the expiration time $T$, the lessee stops payments to the lessor and terminates the lease. If the asset is still working at time $T$, then its residual value $x$ goes to the lessor. This case is different from the first one in that at a certain time $T$ the lease must end. The lease may even terminate earlier if the asset is not performing. The differential equation pertaining to this situation is the same,

$$\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - \alpha\frac{\partial V}{\partial x} - rV = -\frac{\partial V}{\partial t} - p \tag{13}$$

but the boundary conditions are different.

First, change the variable $t$ and let $\tau = -t$ such that $\tau = T$ at the beginning of the lease and $\tau = 0$ at the termination of the lease. Equation (13) may then be written as

$$\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - \alpha\frac{\partial V}{\partial x} - rV = \frac{\partial V}{\partial \tau} - p \tag{30}$$

When the lease finishes at $\tau = 0$ with the asset having a value of $x$, then the lease value $V(x,\tau)$ is given by

$$V(x,0) = x,\ x \geq 0 \tag{31}$$

Whenever the asset value becomes zero, the lease terminates and its value becomes zero. This gives

$$V(0,\tau) = 0 \tag{32}$$

It is also required that the rate of change of $V$ with respect to $x$ remains finite as $x$ increases indefinitely, as given by (18). To evaluate this lease, solve (30) with the boundary conditions (18), (31) and (32). The result is

$$\begin{aligned} V_2(x,\tau) = &-\frac{p}{r}\exp\left(\frac{x(\alpha - \sqrt{\alpha^2 + 2r\sigma^2})}{\sigma^2}\right)\mathrm{N}\left(\frac{-x + \tau\sqrt{\alpha^2 + 2r\sigma^2}}{\sigma\sqrt{\tau}}\right) \\ &-\frac{p}{r}\exp\left(\frac{x(\alpha + \sqrt{\alpha^2 + 2r\sigma^2})}{\sigma^2}\right)\mathrm{N}\left(\frac{-x - \tau\sqrt{\alpha^2 + 2r\sigma^2}}{\sigma\sqrt{\tau}}\right) \\ &+ (x + \alpha\tau + \frac{p}{r})\exp\left(\frac{2\alpha x}{\sigma^2} - r\tau\right)\mathrm{N}\left(\frac{-x - \alpha\tau}{\sigma\sqrt{\tau}}\right) \\ &+ (-x + \alpha\tau + \frac{p}{r})\, e^{-r\tau}\,\mathrm{N}\left(\frac{-x + \alpha\tau}{\sigma\sqrt{\tau}}\right) + (x - \alpha\tau - \frac{p}{r})\, e^{-r\tau} + \frac{p}{r} \end{aligned} \tag{33}$$

where N(·) is the cumulative normal density function defined by

$$\mathrm{N}(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}\, dt \tag{34}$$

As noted earlier, if $x = 0$ then $V = 0$. The value of the lease drops to zero when the value of the underlying asset becomes zero. If $\tau = 0$ then $V = x$. At expiration, the value of the lease is equal to the value of the underlying asset.

Putting $p = 0$ in (33) gives

$$V_2(x,\tau,p\!=\!0) = \frac{r(x+\alpha\tau)}{r}\exp\!\left(\frac{2\alpha x}{\sigma^2} - r\tau\right) \mathrm{N}\!\left(\frac{-x-\alpha\tau}{\sigma\sqrt{\tau}}\right) - \frac{r(x-\alpha\tau)}{r}\, e^{-r\tau}\left[1 - \mathrm{N}\!\left(\frac{-x+\alpha\tau}{\sigma\sqrt{\tau}}\right)\right] \qquad (35)$$

This represents the value of a lease on an asset which is expected to remain unleased for time $\tau$. The lease generates no revenue while the asset is depreciating steadily. The value of the lease is entirely due to the residual value of the asset after time $\tau$.

The main result of this section, equation (33), gives the value of a lease of an asset whose value is depreciating linearly but in a stochastic manner. It is assumed that the straight-line method of depreciation represents the correct value of the asset with time.

Assume that the value of the asset $x$ is \$100 and that it has a lease for a time $\tau$ of 5 years. It is depreciating at the rate $\alpha$ of \$20 per year with a standard deviation $\sigma$ of \$5 per year$^{1/2}$. The lease payment $p$ is \$25 per year, paid continuously. The relevant discount rate is 10%. Substituting these values in (33), the value of the lease is found to be \$97.58. The net present value of the decision to buy the asset for \$100 and then lease it for 5 years is thus –\$2.42. If the lease were infinitely long, its value would be \$98.13 as seen previously in Case 1.

The calculation of lease value uses different asset values and different interest rates. Figure 4 shows the results in graphic form where $x$ = \$70 to \$100, $r$ = 0 to 0.2 per year, $p$ = \$25 per year, $\tau$ = 5 years, $\alpha$ = \$20 per year, and $\sigma$ = \$5 per year$^{1/2}$. Because the life of the lease is limited, these curves are lower than the ones in Figure 1 which shows the leases with unlimited time.

**Figure 4: Lease Value and Discount Rates**

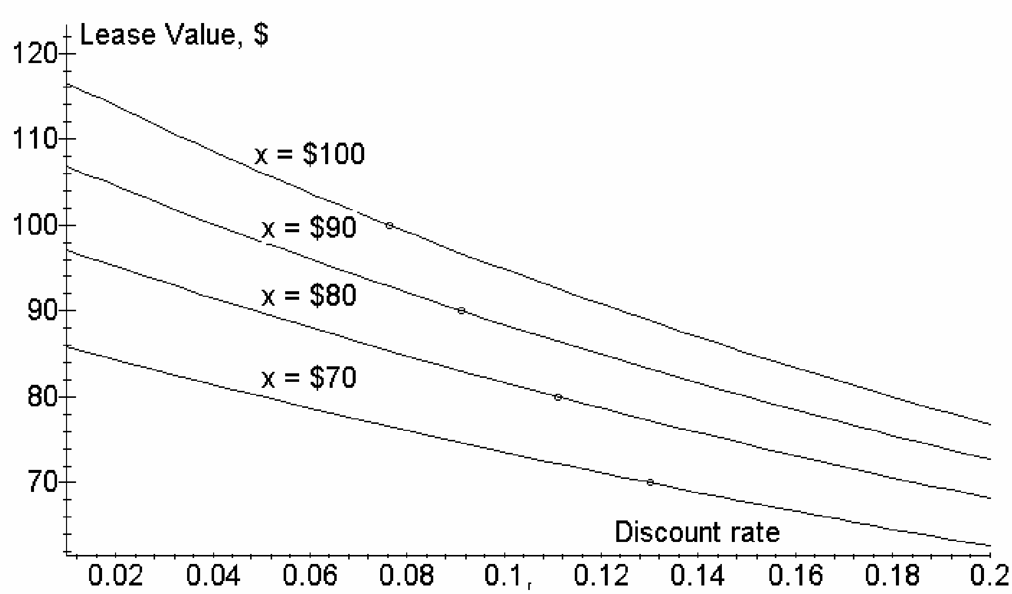


**Figure 4. The diagram above shows how the value of a 5-year lease with asset values of \$70, \$80, \$90, and \$100 ~~at~~ varies with different discount rates. The other parameters are $p$ = 25, $\alpha$ = 20, $\sigma$ = 5. The small circles on the curves represent the points where the NPV of the lease is zero for the lessor.**

Increasing the rate of depreciation will lower the value of the lease, as seen in Figure 5. The curves are for asset values from \$70 to \$100 for five-year leases with depreciation ranging from \$0 to \$40 per year. The annual lease payment $p$ is \$25 annually, paid continuously. Other parameters are $r$ = 0.1 per annum and $\sigma$ = \$5 per year$^{1/2}$.

**Figure 5: Lease Valuation versus Depreciation**

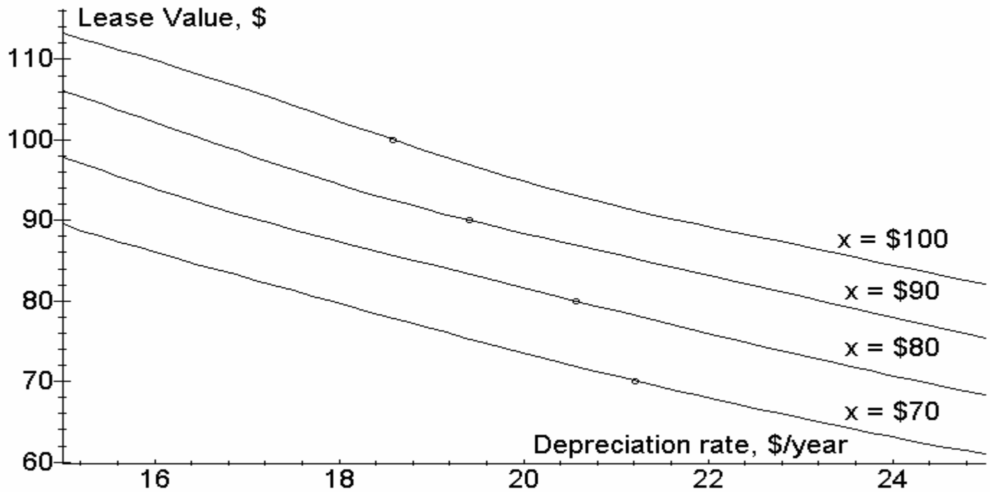


**Figure 5: The diagram above shows how the value of a five-year lease on assets with initial values from \$70 to \$100 varies with depreciation rates from \$15 to \$25 per year. The small circles on the curves represent the points where the NPV of the lease is zero for the lessor.**

**Figure 6: Lease Valuation**

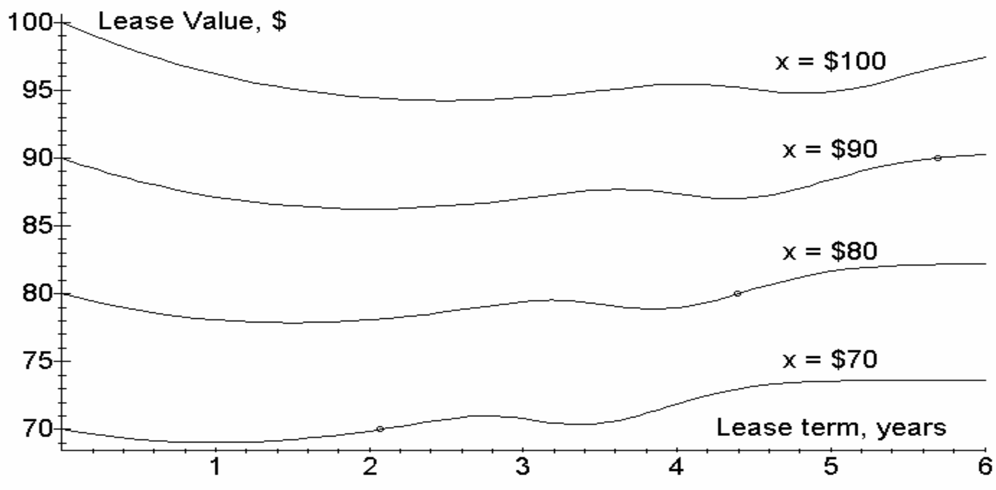


**Figure 6: The diagram above shows how the value of leased assets with initial values from \$70 to \$100 vary with changes in the lease term for up to 6 years. The other parameters are $\alpha$ = 20, $\sigma$ = 5, and $r$ = .1. The small circles on the curves represent the points where the NPV of the lease is zero for the lessor.**

The value of the lease also depends upon the term of the lease. The lease value shows two local minima at different times. The lease value levels off at longer times. The results are presented in Figure 6 which uses the parameters $x$ = \$70 to \$100, $\tau$ = 0 to 6 years, $p$ = \$25 per year, $r$ = 0.1 per year, $\alpha$ = \$20 per year, and $\sigma$ = \$5 per year$^{1/2}$.

The model presented here resembles the Black-Scholes option-pricing model. It starts by using similar basic assumptions. However, it is different from Black-Scholes in at least three ways. First, it assumes that the process follows *arithmetic* Brownian motion which is reasonable for an asset depreciating linearly with age in contrast to the Black-Scholes assumption of *geometric* Brownian motion which is appropriate for the price of a security trading in the markets.

Secondly, this model assumes a continuous cash flow from the lease. The Black-Scholes model specifies that the cash flows from the security, namely dividends, are zero. The use of cash flows in this case gives rise to the particular solution of the problem. Third, it is assumed that the assets are not infinitely divisible, and that it is not possible to form a risk-free hedge by continuously adjusting the assets and their leases.

## 5. Case 3

The lease in this third case is similar to the one in Case 2 but with the additional clause that the lessee has the right, but not the obligation, to buy the asset at the end of the contract period by making a payment $M$ to the lessor. If the value of the asset is more than $M$ at the termination of the lease, the lessee will exercise the option to buy; and this will limit the maximum value of the lease at termination to be $M$ for the lessor.

The basic differential equation to solve is the same,

$$\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - \alpha\frac{\partial V}{\partial x} - rV = \frac{\partial V}{\partial \tau} - p \tag{30}$$

The boundary conditions are, however, different. The first boundary condition is on $\tau$, the time remaining in the lease. When the lease expires at $\tau = 0$, the value of the lease is given as

$$V(x,0) = x, \qquad 0 \le x < M \tag{36}$$
$$= M, \qquad M \le x < \infty \tag{37}$$

This occurs because the lessee has the right to buy the asset by paying at most $M$ to the lessor. If the lessee does not exercise the option to buy the asset, then the lessor receives the asset, whose residual value is $x$. The second boundary condition is on the value of the asset $x$. When $x$ becomes zero, the value of the lease must also become zero, or

$$V(0,\tau) = 0 \tag{38}$$

The solution of (30) subject to boundary conditions (36-38) is given by

$$V_3(x,M,\tau) = V_2(x,\tau)$$
$$- (x - M + \alpha\tau)\, e^{-r\tau}\, \mathrm{N}\left(\frac{-M + x - \alpha\tau}{\sigma\sqrt{\tau}}\right) - (x + M + \alpha\tau)\exp\left(\frac{2\alpha x}{\sigma^2} - r\tau\right)\mathrm{N}\left(\frac{-x - M - \alpha\tau}{\sigma\sqrt{\tau}}\right)$$
$$- \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}}\left[\exp\left(\frac{\alpha(x - M - \alpha\tau/2)}{\sigma^2} - \frac{(x-M)^2}{2\sigma^2\tau} - r\tau\right) - \exp\left(\frac{\alpha(x - M - \alpha\tau/2)}{\sigma^2} - \frac{(x+M)^2}{2\sigma^2\tau} - r\tau\right)\right] \quad (39)$$

where $V_2(x,\tau)$ is given by (33).

To check the validity of the solution, note that (39) satisfies the basic differential equation (30). It also satisfies the boundary conditions as follows. When $x = 0$, the value of the lease is also zero, $V = 0$. When the lease expires at $\tau = 0$, the value of the lease is $x$ if $x < M$, otherwise it is $M$. Suppose the lessee does not have the option to buy the asset at the end of the lease. This is similar to the purchase price being infinite. Substituting $M = \infty$ in (39), the value reduces to that of Case 2, (33) where

$$\underset{M \to \infty}{limit}\; V_3(x,\tau) = V_2(x,\tau) \tag{40}$$

Assume that the lease has an infinitely long term, that is, $\tau = \infty$. Substituting this value in (39), it reduces to the result of the Case 1, (18).

$$\underset{\substack{M \to \infty \\ \tau \to \infty}}{limit}\; V_3(x,\tau) = V_1(x) \tag{41}$$

Suppose the ownership of the asset is simply turned over to the lessee at the end of the lease, that is, $M = 0$. This is similar to an installment sale where the seller guarantees that the asset will function properly during the installment-payment period. However, if the asset breaks down during this period, then the buyer can stop further payments on the purchase. The value of the lease in this case is

$$V_3(x,0,\tau) = -\frac{p}{r}\exp\left(\frac{x\,(\alpha + \sqrt{\alpha^2 + 2r\sigma^2})}{\sigma^2}\right)\left[1 - \mathrm{N}\left(\frac{x + \tau\sqrt{\alpha^2 + 2r\sigma^2}}{\sigma\sqrt{\tau}}\right)\right]$$
$$- \frac{p}{r}\exp\left(\frac{x\,(\alpha - \sqrt{\alpha^2 + 2r\sigma^2})}{\sigma^2}\right)\left[1 - \mathrm{N}\left(\frac{x - \tau\sqrt{\alpha^2 + 2r\sigma^2}}{\sigma\sqrt{\tau}}\right)\right] + \frac{p}{r}(1 - e^{-r\tau})$$
$$+ \frac{p}{r}\exp\left(\frac{2\alpha x}{\sigma^2} - r\tau\right)\left[1 - \mathrm{N}\left(\frac{x + \alpha\tau}{\sigma\sqrt{\tau}}\right)\right] + \frac{p}{r}e^{-r\tau}\left[1 - \mathrm{N}\left(\frac{x - \alpha\tau}{\sigma\sqrt{\tau}}\right)\right] \tag{42}$$

The main result of this section is (39), which represents the value of a lease from the lessor's point of view of an asset depreciating steadily in value when the lessee has the option to purchase the asset for a fixed price at the termination of the lease. The two previous results, (19) and (33), turn out to be the special cases of the general formula.

To get a feel for the result, numerical values for various parameters can be substituted as follows. Let the initial value of the asset $x$ be \$100. The term of the lease contract $\tau$ is 5 years. However, the lease may terminate earlier if the asset value falls to zero and no further lease payments are due. The linear rate of depreciation of the asset $\alpha$ is \$20 per year. It is again assumed that the straight-line method of depreciation describes the economic value of the asset. The standard deviation of the rate of depreciation $\sigma$ is \$5 per year$^{1/2}$. The proper discount rate $r$ is 10%. Assume a constant interest rate or that the yield curve is flat. The final purchase price of the asset $M$, if the lessee elects to buy the asset at the termination of the lease, is \$10. Assume that the lease payment $p$, paid continuously, is \$25 per year.

Substituting the numbers into (39) the value of the lease is \$96.89. Thus, if the lessor buys the asset for \$100, the net present value of the buy-and-lease decision is –\$3.11. This value is slightly less than the answer found in Case 2, \$96.58. The difference, \$0.69, represents the value of the option available to the lessee. The value of the lease is now lower for the lessor.

The maximum value of the lease can now be found by calculating the present value of the lease payments and the final purchase price to get

$$V_{\max} = \int_0^T p\, e^{-rt}\, dt + M\, e^{-rT} = \frac{p}{r}\,(1 - e^{-rT}) + M\, e^{-rT} \tag{43}$$

Again, substituting the numbers in the above expression gives $V_{\max}$ = \$104.33. This implies that it is unprofitable for the lessor to lease out an asset with a value more than \$104.33 under the terms of the above lease because the maximum amount recoverable through the lease is \$104.33.

**Figure 7: Lease Valuation With Option to Buy**

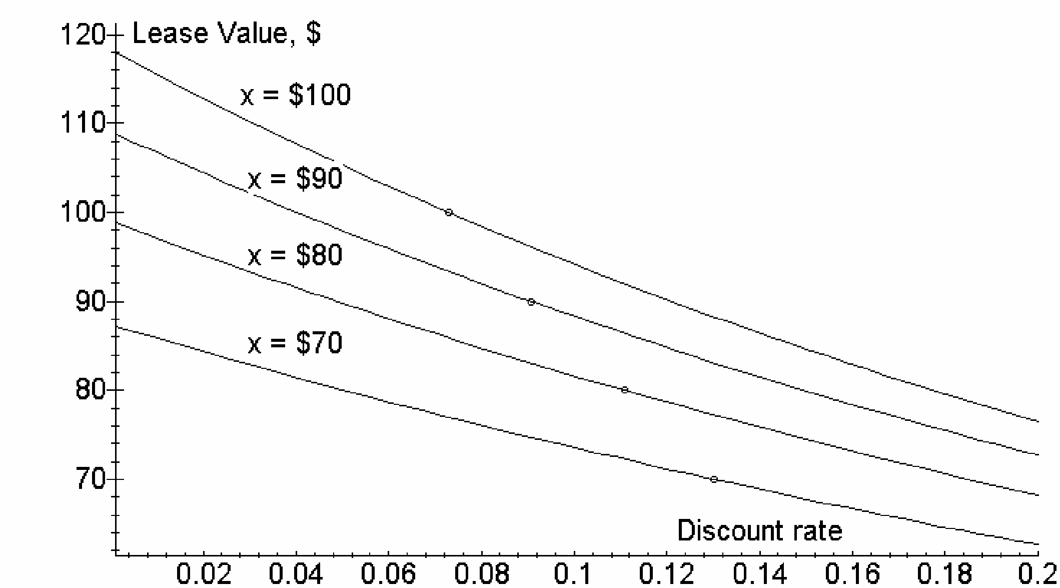


**Figure 7: The diagram above shows how the value of a lease with asset values from \$70 to \$100 changes with the discount rate when the lessee has the option to buy the asset for \$10 when the lease expires after 5 years. Other parameters are $\alpha$ = 20, $\sigma$ = 5, and $p$ = 25. The small circles on the curves represent the points where the NPV of the lease is zero for the lessor.**

The value of a lease is expected to decrease due to rising interest rates because the present value of the lease payments and the final purchase price become lower at a higher discount rate. Figure 7 illustrates the value of a lease on assets with different initial values.

**Figure 8: Lease Valuation versus Depreciation Rate**

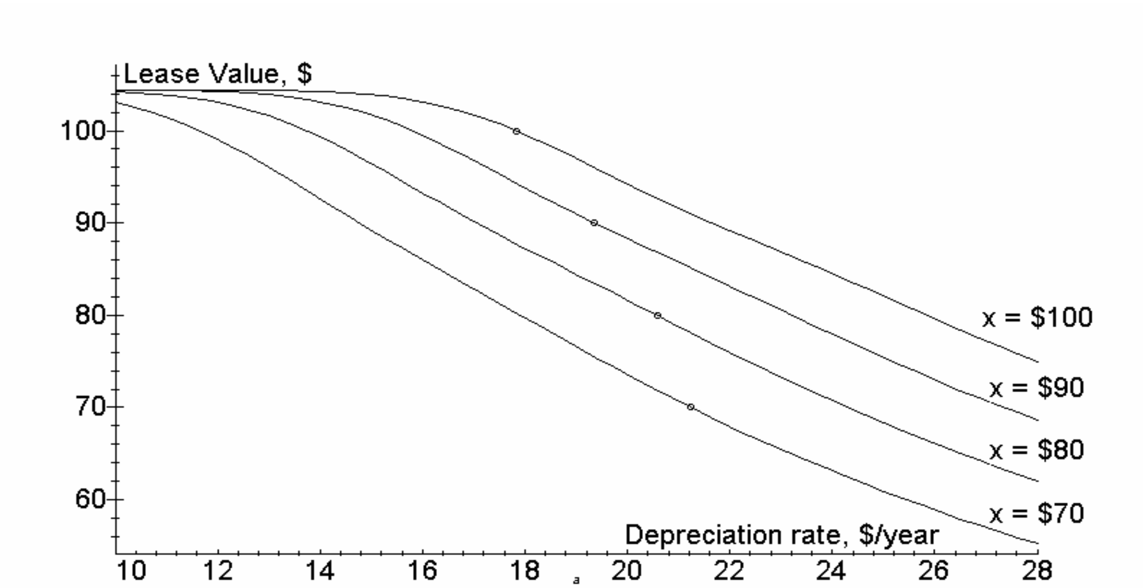


**Figure 8: The diagram above shows how lease values with a five-year term on assets with values from $70 to $100 changes with various depreciation rates ranging from $10 to $30 per year. The lessee has the option to buy the asset for $10 at the end of the lease. Other parameters are *p* = 25, and σ = 5. The small circles on the curves represent the points where the NPV of the lease is zero for the lessor.**

Next to consider is the effect of the rate of depreciation of the asset. If the asset is depreciating rapidly, it may not be functioning at all by the end of the lease period. This will reduce the value of the lease substantially. On the other hand, an asset that is depreciating very slowly will still be somewhat valuable at the end of the lease. For such an asset, the lessor should expect to receive the lease payments and the purchase price at the end of the lease. Using (43), the value of such a lease is $104.33. Figure 8 represents these results graphically.

Figure 9 shows that the value of a lease increases with the term of the lease. The lease value reaches a plateau for a very long lease period because the asset will eventually break down and the lease will become void. In such a case the maximum value of the lease is found by setting $\tau = \infty$ in (39). After some simplification, this yields the same result (19) as in case 1.

$$V_{\max}(x) = \frac{p}{r}\left[1 - \exp\left(\frac{x[\alpha - \sqrt{\alpha^2 + 2\sigma^2 r}]}{\sigma^2}\right)\right] \tag{19}$$

Equation (41) gives the maximum value of the lease for an asset with no variance in its depreciation rate. For such an asset, the lessor will receive the lease payments and the final purchase price.

**Figure 9: Asset Value, Lease Term, and Lease Value**

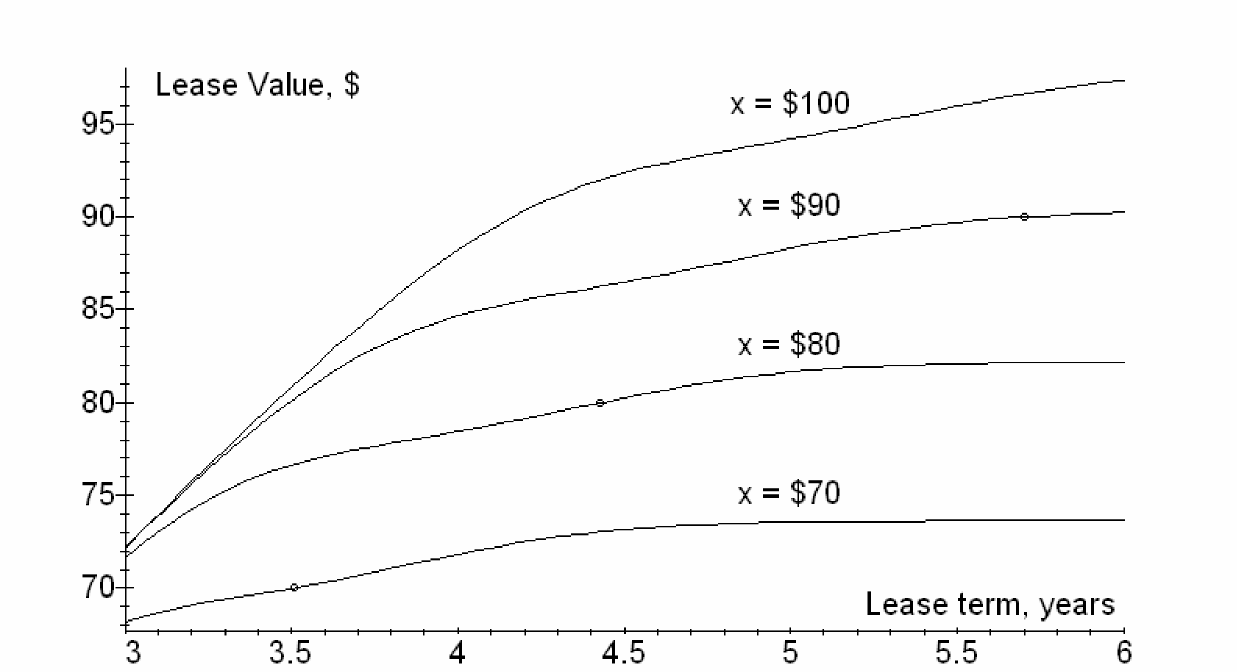


**Figure 9: The diagram above shows how the lease values increase with the asset values *x* and the term of the lease. The asset values range from $70 to $100, and the lease term ranges from 3 to 6 years. The other parameters are $\alpha$ = 20, *r* = .1, $\sigma$ = 5, and *p* = 25. The small circles on the curves represent the points where the NPV of the lease is zero for the lessor.**

One might expect that a higher final purchase price would have a positive impact on the value of the lease. However, in this particular case the impact is not that dramatic because the asset value is almost zero at the end of 5 years. The minimum value of the lease will occur in the case when the asset is simply given away to the lessee at the end of the lease, that is, when $M = 0$. The lessor will receive only the lease payments with the possibility of early termination of the lease due to the asset value dropping to zero. Calculated from (42), this value is $94.89. At the other extreme, if $M = \infty$, it produces the previous result from Case 2, (31). The numerical value of the lease in this case is then $97.58.

## 6. Conclusions

The results of the above analysis can be used to find the values of some real leases. Consider an automobile dealer who can sell a new car for $20,000 to a customer. The customer may arrange financing for three years at a 9% interest rate with a monthly installment of $635.99. The car dealer may lease it to the customer for $550 a month for the next 36 months. Assume the car depreciates in value at the rate of $4000 annually with a standard deviation of $1000 per year$^{1/2}$. The customer can buy the car after 36 months for some unknown price. The car dealer can calculate the unknown price such that leasing and selling the car become equivalent by substituting $x$ = \$20,000, $r$ = .09 per year, $\tau$ = 3 years, $\alpha$ = \$4000 per year, $\sigma$ = 1000 per year$^{1/2}$, $p$ = \$6600 per year into (41). The dealer will then break even if he gives an option to the customer to buy the car for \$3471 at the end of the lease.

Suppose a person owns a house with current value equal to \$100,000. The owner expects the house to appreciate by \$4000 annually for the next two years with a standard deviation of \$3000. Assume the house is rented to a tenant for two years with an annual rent of \$12,000 paid continuously, after covering all expenses. Assume the discount rate is 8%. The tenant also wants an option to buy the house at the end of the lease. The owner will break even if he gives a free option with an exercise price of \$94,695.

For future research, it would be desirable to perform an empirical analysis of these results using actual data for car resale values. One could also verify the results by using data for home values. It would also be possible to extend the results by including the tax rates of the lessor and the lessee in the analyses.

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